ESTIMATING UNCERTAINTIES in ANTENNA MEASUREMENTS

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Michael H. Francis

National Institute of Standards and Technology
325 Broadway
Boulder, CO 80305-3328 USA
francis@boulder/nist.gov
1-303-497-5873

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INTRODUCTION

Why do uncertainty analysis?

In general, the result of a measurement is only an estimate of the value of the quantity subject to measurement (the measurand) and the result is complete only when it is accompanied by a quantitative statement of its uncertainty.

$$A = A_0 \pm \Delta A,$$

where A_0 is the best estimate and ΔA is the uncertainty.

The error is the difference between the true value and the measured value. The true value is unknowable so the error is unknowable.

$$Error = A_0 - A_t$$

The uncertainty is the spread of values in which the true value may reasonably be expected to lie.

SOURCES of UNCERTAINTY in ANTENNA MEASUREMENTS

- A good uncertainty analysis requires that all major sources of uncertainty be identified.

Probe/reference antenna properties
Alignment/positioning
Receive system
Drift
Impedance mismatch
Leakage and crosstalk
Flexing cables
Multipath
Normalization
Non-uniform illumination

Aliasing
Measurement area truncation
Antenna-antenna multiple reflections
Random errors (e.g., noise)

DISTRIBUTING ERRORS (ISO)

The ISO approach is to assume that by some means we have knowledge of how the errors are distributed. Let us note some properties of distribution functions.

1. The variables x_1, x_2, …, x_n are independent only if the distribution function is separable, i.e.,

$$f(\underline{x}) = f(x_1, x_2, \ldots, x_n) = f_1(x_1) f_2(x_2) \ldots f_n(x_n) = \prod_{i=1}^{n} f_i(x_i).$$

2. The expectation value E of any function $g(\underline{x}) = g(x_1,x_2,\ldots,x_n)$ is

$$E(g(\underline{x})) = \int \ldots \int g(\underline{x}) f_1(x_1) f_2(x_2) \ldots f_n(x_n) dx_1 dx_2 \ldots dx_n$$

Mean and Variance

The mean and variance of a linear function $y = \sum_{i=1}^{n} x_i$ are

$$\mu_y = \sum_{i=1}^{n} \mu_i \qquad (1)$$

and

$$\sigma_y^2 = \sum_{i=1}^{n} \sigma_i^2 \qquad (2)$$

To make use of (1) and (2) for cases where y cannot be written as a sum of other functions, we must linearize the problem by using the first two terms of the Taylor series expansion. This will give us the familiar form for

$$\sigma_y^2 = \sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i} \right)^2_{\underline{x}=\underline{\mu}} \sigma_{x_i}^2 \qquad (3)$$

where the subscript $\underline{x} = \underline{\mu}$ implies the derivative is evaluated at $x_1 = \mu_{x_1}, x_2 = \mu_{x_2}, \ldots, x_n = \mu_{x_n}$.

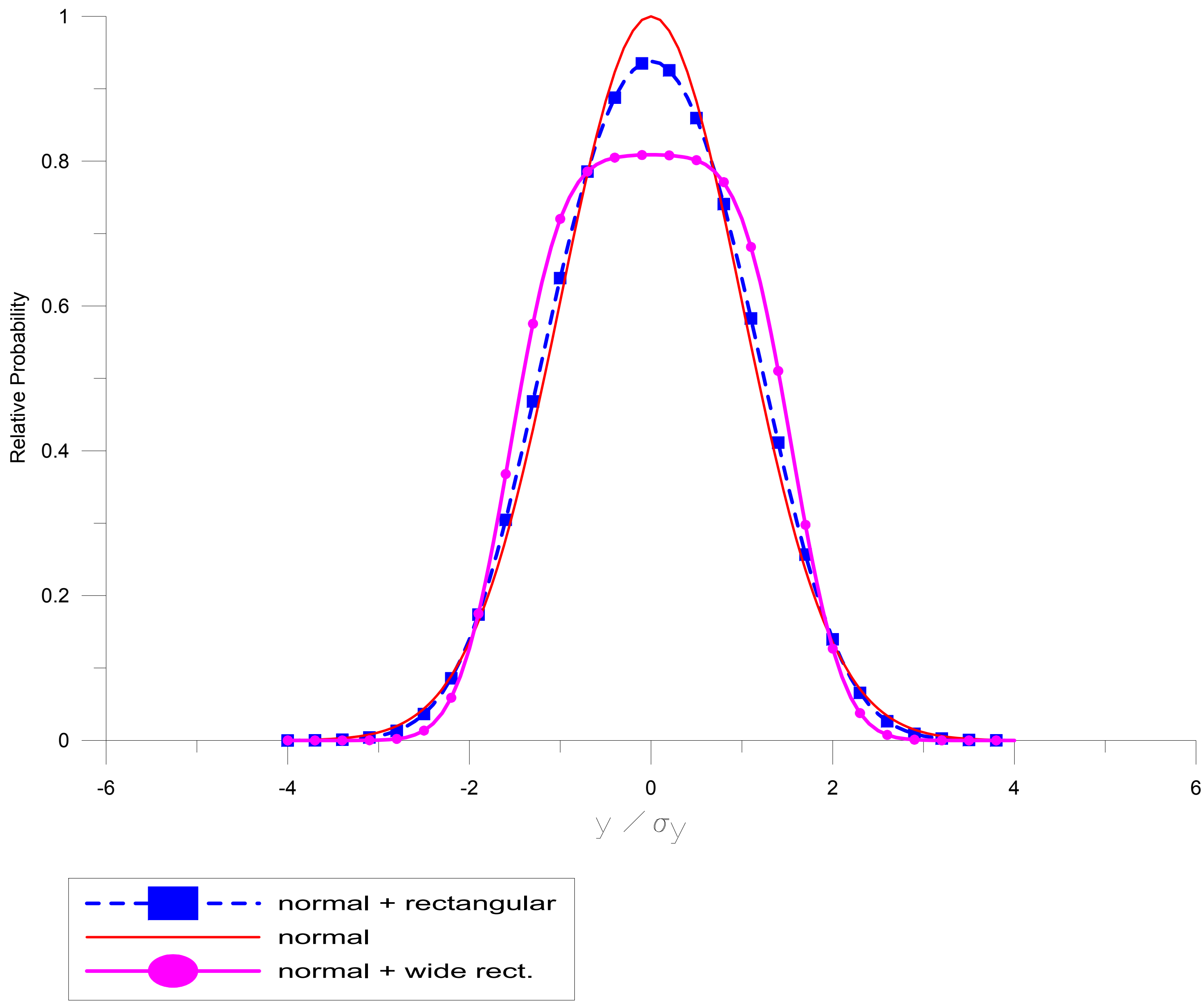
If the errors are large, one might be tempted to use the next term in the Taylor series. This is often problematic since the next term in the Taylor series requires higher moments (see Appendix C), which may not be well known, especially type B uncertainties.

In such cases, it usually makes sense to work directly with y rather than its Taylor series expansion.

Central Limit Theorem

It is often mistakenly asserted that if the number of error sources $n \to \infty$ then the distribution function approaches a normal distribution. The following examples show that this is not a sufficient condition.

First, consider the function $y = \sum_{i=1}^{101} x_i$ with distribution functions $f_1(x_1)\ldots f_{101}(x_{101})$, with $f_1 = \ldots = f_{100}$ = the normal distribution with $\mu_1 = \ldots = \mu_{100} = 0$ and $\sigma_1 = \ldots = \sigma_{100} = \sigma_a$. f_{101} = the rectangular distribution with $\mu_{101} = 0$, and $\sigma_{101} = 10\sigma_a$. As expected, $\sigma_y^2 = \sum_{i=1}^{101} \sigma_i^2 = 200\sigma_a^2$ but the resulting distribution (see graph) is not a normal distribution. Second, consider $\sigma_{101} = 20\sigma_a$ with $\sigma_y^2 = 500\sigma_a^2$. We have plotted these 2 curves plus a normal distribution against y/σ_y so they have the same normalized area and variance.



Relative Probability
1
0.8
0.6
0.4
0.2
0
-6
-4
-2
0
2
4
6
y / σy
normal + rectangular
normal
normal + wide rect.

What is the problem?

We have not satisfied the full set of criteria required for the central limit theorem to be valid! These criteria are

1. $y = \sum_{i=1}^{n} x_i$,
2. x_i are random independent variables with distributions $f_i(x_i)$,
3. for all x_i, $var(x_i)$ is bounded,
4. $n \to \infty$,
5. $\lim_{n\to\infty} var(y) \to \infty$,

then the limiting distribution of $f\left(\dfrac{y-\mu_y}{\sigma_y}\right)$ is the standard normal distribution.

Criteria 3, 4, and 5 imply that $var(y) >> var(x_i)$ for all x_i. It is this requirement that is not satisfied in the above examples.

Another example that does not satisfy the above criteria is:

$$y = \sum_{i=1}^{n} x_i, \text{ and } var(x_i) = \frac{\sigma_0^2}{2^i},$$

Why?

The second order Taylor series also does not satisfy the requirements. Why?

One consequence of these requirements:

If there is a dominant source of uncertainty that is not normally distributed, then it is difficult to associate a confidence level with the value of σ.

TYPE A and TYPE B UNCERTAINTIES

Type A uncertainties are those that can be estimated using statistics (large number of measurements).

Type B uncertainties are those that are estimated in any other way.

The ISO guide indicates that confidence levels should not be associated with the overall uncertainty in the presence of type B uncertainties.

ISO Method of Combining Uncertainties

We combine uncertainties as follows.

For type A uncertainties the overall uncertainty is

$$u_A = \sqrt{\sum_{i=1}^{n_1} u_i^2}, \tag{4}$$

where u_i is the standard deviation of the i^{th} component.

For type B uncertainties

$$u_B = \sqrt{\sum_{j=1}^{n_2} u_j^2}, \tag{5}$$

where u_j is an *estimate* of the standard deviation for the distribution function for the j^{th} component.

The overall uncertainty is

$$u_c = \sqrt{u_A^2 + u_B^2}. \qquad (6)$$

The expanded uncertainty is

$$U = Ku_c, \qquad (7)$$

where K is a coverage factor typically with a value of 2 or 3. $K = 2$ is the value normally used in international intercomparison measurements.

Geometric Motivation for Combining Uncertainties

In the real world, we sometimes have large errors but poor information about the distribution of the errors and we can sometimes only estimate an approximate upperbound error.

If the errors are independent, then we can visualize that they are orthogonal vectors in *n*-space. The resultant vector is the root sum square (RSS) of the individual components.

We would not associate a confidence level with this result (or any type B uncertainty).

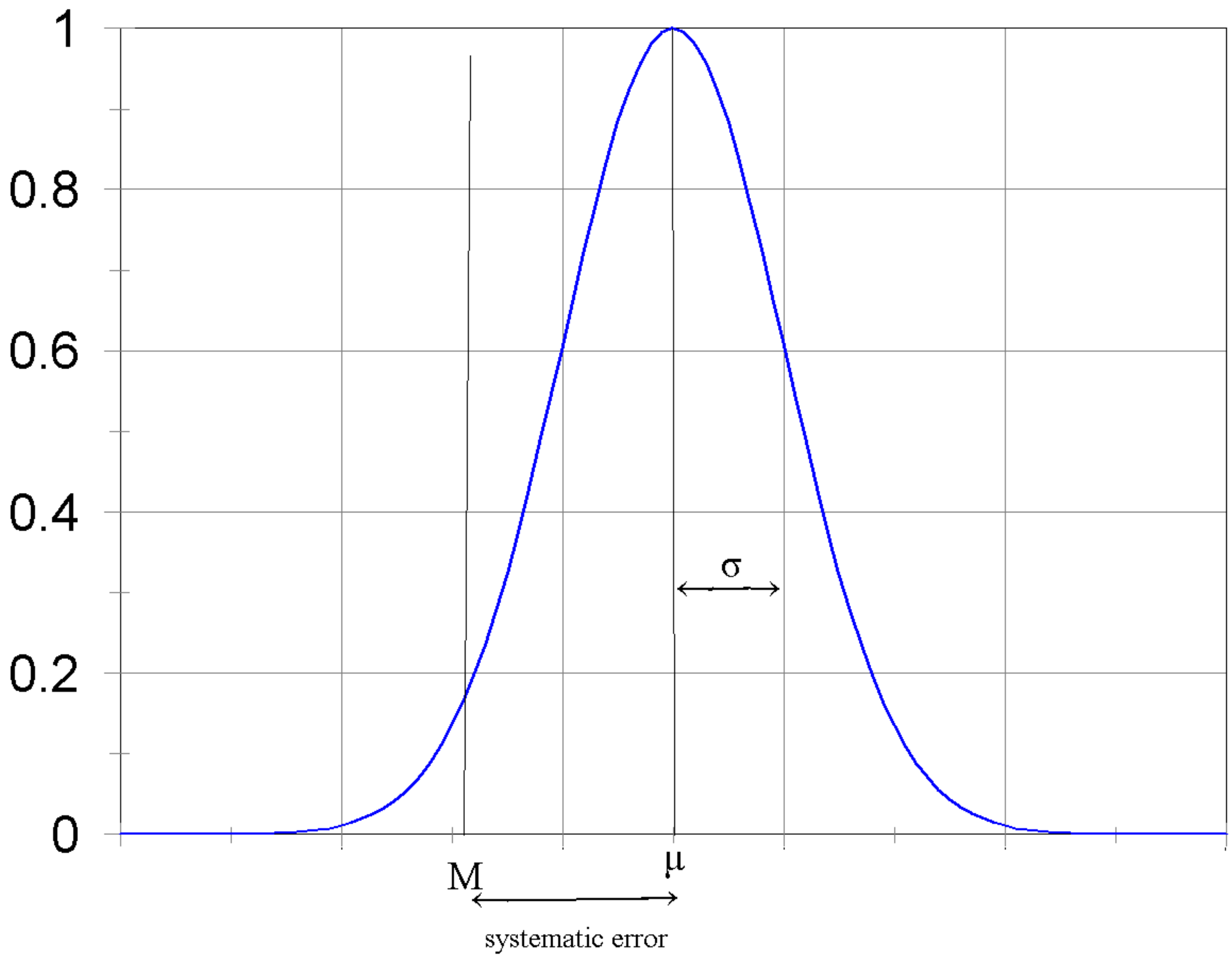
Systematic and Random Errors

Random error

ISO definition: The result of a measurement minus the mean that would result from an infinite number of measurements of the same measurand carried out under repeatability conditions. [Compare this to a common definition of: "uncorrelated from measurement to measurement."]

Systematic error

The mean that would result from an infinite number of measurements of the same measurand carried out under repeatability conditions minus the true value of the measurand.


1
0.8
0.6
0.4
0.2
0
σ
M
μ
systematic error

Corrected Measurement Result

The corrected (for systematic error) result of the measurement gives the following estimate of the measurand V_M

$$V_M = \mu - e_s \pm_{\sigma^-}^{\sigma^+}, \tag{8}$$

where e_s is the best estimate of the systematic error.

METHODS OF ESTIMATING UNCERTAINTIES

1. Analytical Method

2. Simulation Method

3. Self-Comparison Method (Measurement)

We want to estimate the individual u_i and u_j in (4) and (5). We do not necessarily need to know $\frac{\partial y}{\partial x_j}$ or σ_{x_j} in (3) to do this. Instead, we may be able to estimate $\left(\frac{\partial y}{\partial x_i}\right)^2 \sigma_{x_i}^2$ directly using one of the methods above.

ANALYTICAL METHOD

We use our knowledge of the theory to develop equations to provide estimates of uncertainty.

Advantages: More generally apply; Valid for many situations; Can identify upper-bound uncertainties.

Limitation: Some estimates, particularly worst-case estimates, of uncertainty are unrealistically high.

Example: In planar near-field measurements, we assume a periodic probe-position error to obtain an equation to estimate the uncertainty due to probe-position errors. This provides an upper-bound error. Periodic position errors cause larger (than non-periodic) errors in the pattern but are concentrated in two directions. Other types of position errors give smaller pattern errors but spread them out over more of the pattern (see Yaghjian).

$$\left|\frac{F_e(\theta,\phi)}{F(\theta,\phi)}\right|_{dB} \leq 13.5\left(\frac{\delta_z}{\lambda}\right)\cos\Theta_B\, g(\theta,\phi),$$

where F_e and F are the pattern with and without errors respectively, $g(\theta,\varphi)$ is the ratio of the peak far-field pattern to the pattern value in the θ,φ direction and is greater than 1. Θ_B is the direction of the peak far-field pattern, and δ_z is the maximum z-position error.

SIMULATION METHOD

We use a simulation to model an error and estimate its effect on the result.

Advantage: Study individual errors; Use measured or ideal data; Can study many cases.

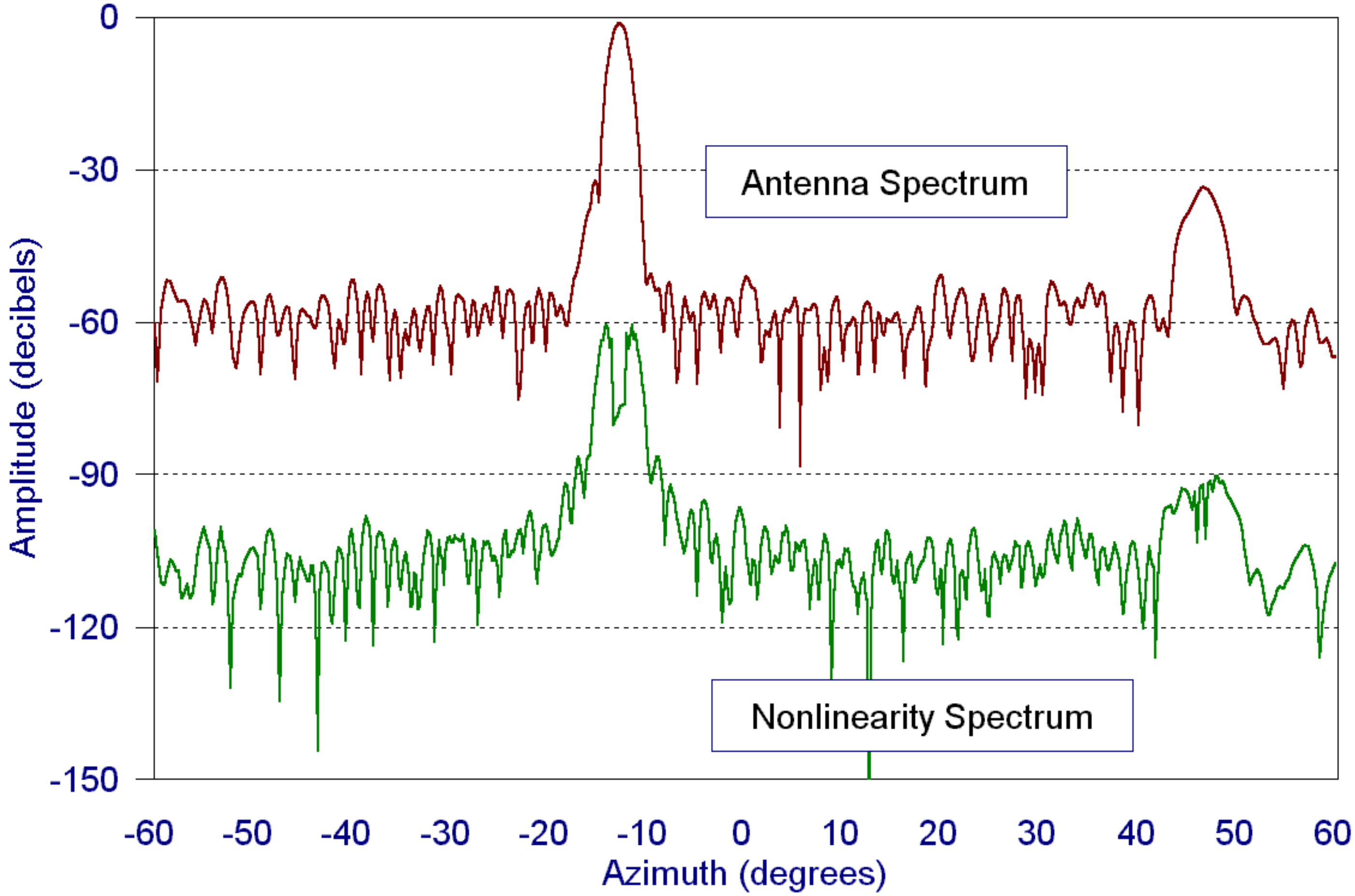
Limitation: Must assume the form of the error; Results are not general.

Example: We model a receiver non-linearity effect by assuming the receiver output has additional terms than the linear term such as

$$\text{Output} = c_1 * \text{input} + c_2 * (\text{input})^2,$$

and then see how this affects the result.

NONLINEARITY TEST RESULTS



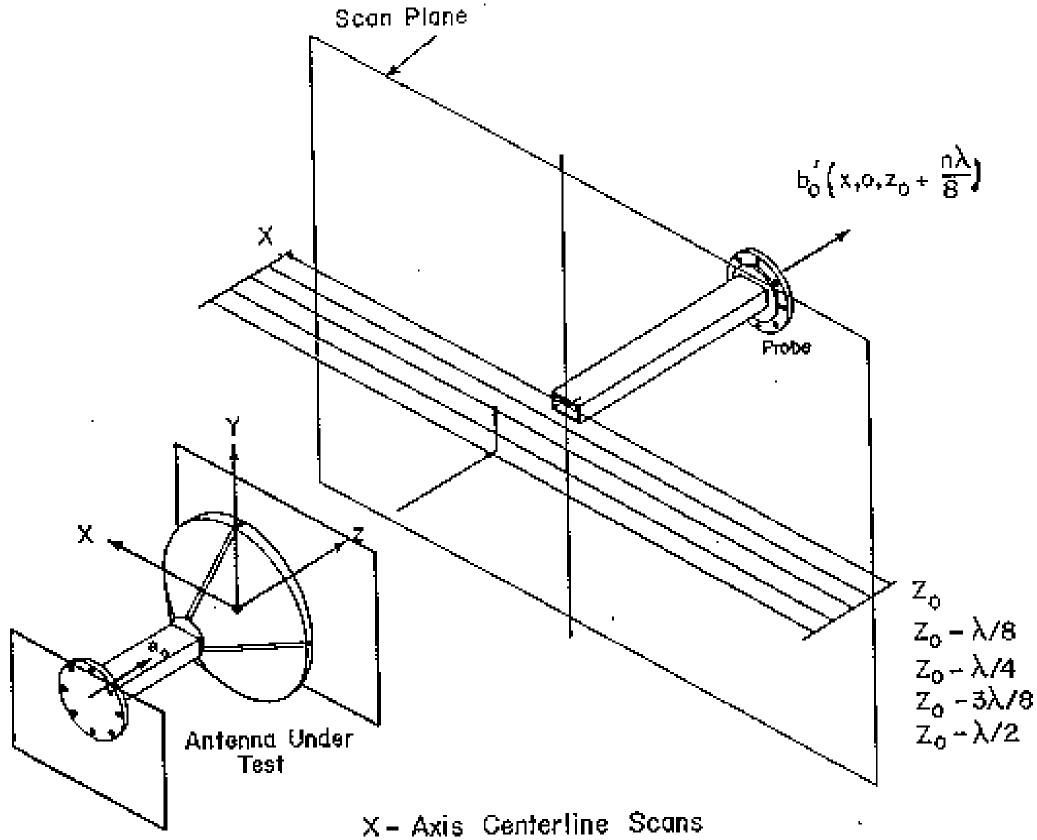
SELF-COMPARISON METHOD

We change the measurement system in a controlled manner to change the error in some known way.

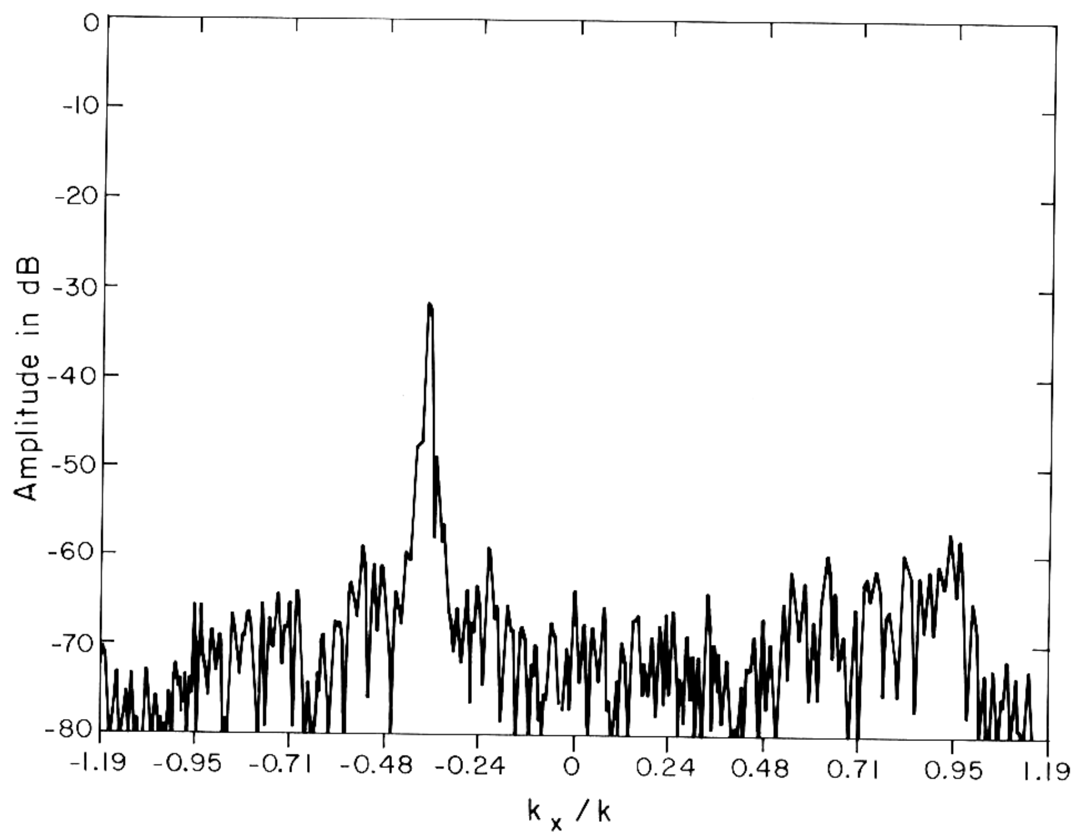
Advantages: Directly measure some uncertainties; Can verify simulation and theory; Applies to actual measurement system.

Limitation: Time consuming; Results are not general.

Example: Multiple reflections between the antenna under test and the probe have period of $\lambda/2$ in the separation distance. By changing the separation distance by $\lambda/4$ we can change the multiple reflections from in phase to out of phase. By comparing the pattern and gain results from both measurements we can estimate the uncertainty due to multiple reflections.



X- Axis Centerline Scans
to Measure Multiple Reflections



0
-10
-20
-30
-40
-50
-60
-70
-80
Amplitude in dB
-1.19
-0.95
-0.71
-0.48
-0.24
0
0.24
0.48
0.71
0.95
1.19
k x / k

EXAMPLE
ULSA ARRAY PATTERN UNCERTAINTIES

UNCERTAINTY SOURCE	UNCERTAINTY LEVEL (Relative to peak)
Antenna-antenna multiple reflections	2.8e-04
Multipath (room scattering)	1.6e-04
Flexing cables	1.6e-04
Alignment / positioning errors	9.0e-05
Leakage	9.0e-05
Noise / Random errors	9.0e-05
Measurement area truncation	1.6e-05
Aliasing	1.6e-05
Receive system non-linearity	5.0e-06
ROOT SUM SQUARE	4.0e-04
Expanded uncertainty (K=2)	7.9e-04

SIDELOBE LEVEL	Expanded Uncertainty in dB	
-30 dB	±.22	
-45 dB	+1.1	-1.3
-55 dB	+3.2	-5.1
-60 dB	+5.1	-13.6
-62 dB	+6.0	- ∞

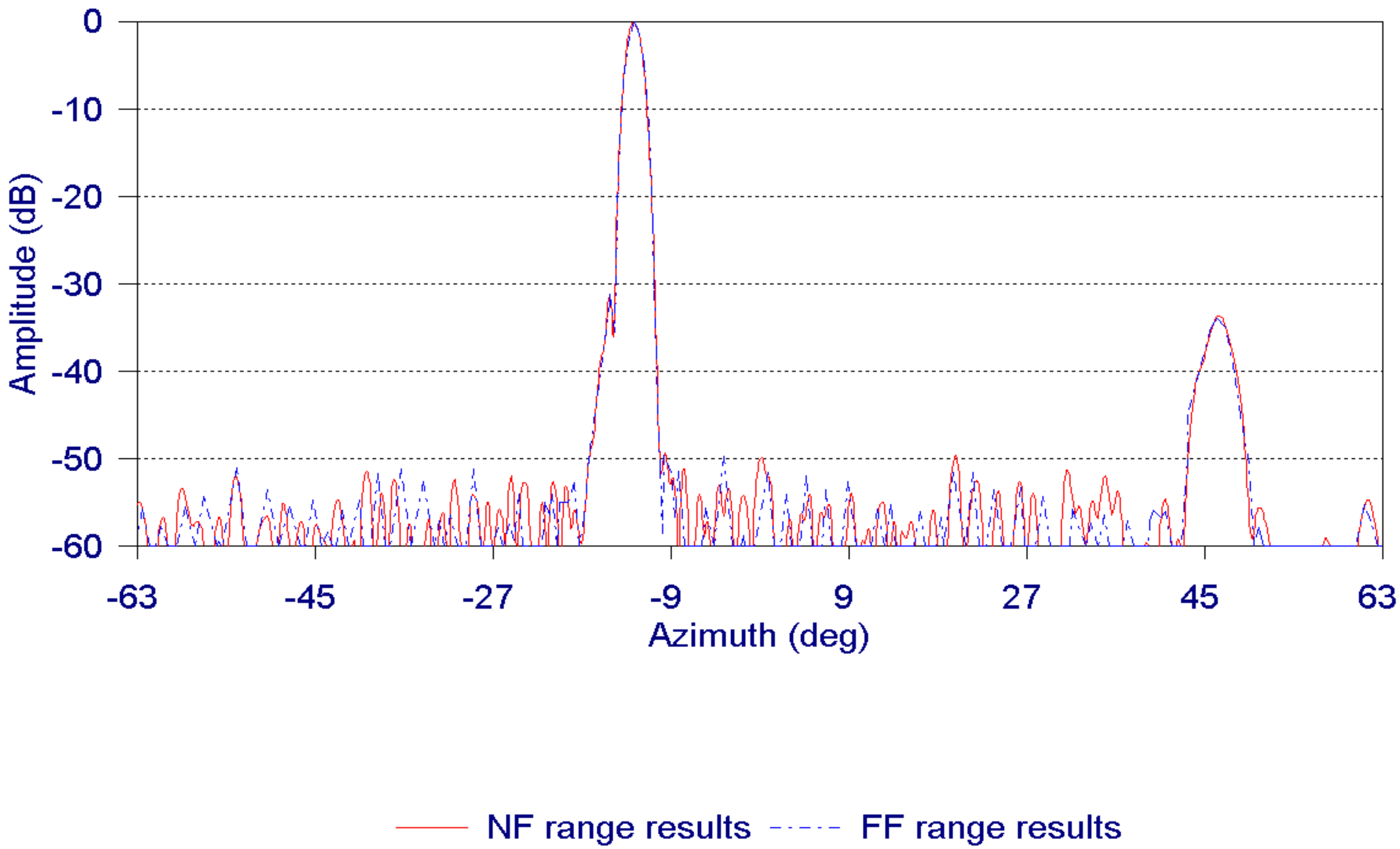
Measurement Intercomparison

Since there can be much subjectivity in evaluating type B uncertainties, some intercomparison measurements are necessary as a reality check!

Measurements from different ranges should agree to within the combined uncertainties. That is, the uncertainty bars should overlap.

Comparison of NF and FF Results

ULSA Array



CONCLUSION

- "Rigorous uncertainty analysis" is an oxymoron! If it were rigorous, it would not be uncertain.

- A measurement is incomplete without an uncertainty analysis.

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APPENDICES

APPENDIX A
Linear Sums of Random Variables

If $y = \sum_{i=1}^{n} a_i x_i$ with distribution functions $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$ then the mean $\mu_y = \sum_{i=1}^{n} \mu_i$ and the variance $\sigma_y^2 = \sum_{i=1}^{n} \sigma_i^2$.

$$\mu_y = \int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} a_i x_i \right) \left(\prod_{i=1}^{n} f_i(x_i)\, dx_i \right)$$

$$= \int_{-\infty}^{\infty} a_1 x_1 f_1(x_1) dx_1 \left(\int_{-\infty}^{\infty} \prod_{i \neq 1} f_i(x_i) dx_i \right)$$

$$+ \int_{-\infty}^{\infty} a_2 x_2 f_2(x_2) dx_2 \left(\int_{-\infty}^{\infty} \prod_{i \neq 2} f_i(x_i) dx_i \right) + \ldots$$

$$\boxed{\mu_y = \sum_{i=1}^{n} \mu_i}$$

$$\sigma_y^2 = \int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n} \mu_{x_i} \right)^2 \left(\prod_{i=1}^{n} f_i(x_i) dx_i \right)$$

$$\sigma_y^2 = \int_{-\infty}^{\infty} \left[\sum_{i=1}^{n} \left(a_i x_i - \mu_i \right)^2 + 2 \sum_{i=1}^{n} \sum_{j \neq i} \left(a_i x_i - \mu_i \right) \left(a_j x_j - \mu_j \right) \right] \left(\prod_{i=1}^{n} f_i(x_i) dx_i \right)$$

The second term in the square brackets is zero when x_i and x_j are independent. Thus,

$$\sigma_y^2 = \int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} \left(a_i x_i - \mu_i \right)^2 \right) \left(\prod_{i=1}^{n} f_i(x_i) dx_i \right) = \sum_{i=1}^{n} \sigma_i^2$$

APPENDIX B
Taylor Series to First Order

If $y = g(x_1,\ldots,x_n)$, $x_1,\ldots,x_n$ are independent random variables with distribution functions $f_1(x_1),\ldots,f_n(x_n)$ and we expand g in a Taylor series to first order, then $\mu_y = g(\mu_{x_1},\ldots,\mu_{x_n})$ and $\sigma_y^2 = \sum_{i=1}^{n}\left(\frac{\partial g}{\partial x_i}\right)^2_{\underline{x}=\underline{\mu}} \sigma_{x_i}^2$.

First, we expand g in a Taylor series about the mean values of x_1, …, x_n, then

$$y(x_1,\ldots,x_n) \approx g(\mu_{x_1},\ldots,\mu_{x_n}) + \sum_{i=1}^{n}\left(\frac{\partial g}{\partial x_i}\right)_{\underline{x}=\underline{\mu}} \left(x_i - \mu_{x_i}\right)$$

Then

$$\mu_y = \int_{-\infty}^{\infty} \left[g(\underline{\mu}) + \sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i} \right)_{\underline{x}=\underline{\mu}} \left(x_i - \mu_{x_i} \right) \right] \left(\prod_{i=1}^{n} f_i(x_i) dx_i \right)$$

Since $E(x_i - \mu_{x_i}) = 0$, it follows

$$\boxed{\mu_y = g(\underline{\mu})}$$

$$\sigma_y^2 = E\left[\left(g(x) - \mu_y \right)^2 \right] = \int_{-\infty}^{\infty} \left[\sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i} \right)^2_{\underline{x}=\underline{\mu}} \left(x_i - \mu_{x_i} \right)^2 \right] \left(\prod_{i=1}^{n} f_i(x_i) dx_i \right)$$

$$\sigma_y^2 = \sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i} \right)^2_{\underline{x}=\underline{\mu}} \sigma_{x_i}^2$$

APPENDIX C
Taylor Series to Second Order

Consider the case of Appendix B but expanding to second order in the Taylor series. Find μ_y and σ_y^2.

$$y \approx g(\underline{\mu}) + \sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i} \right)_{\underline{x}=\underline{\mu}} \left(x_i - \mu_{x_i} \right) +$$

$$\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 g}{\partial x_i \partial x_j} \right)_{\underline{x}=\underline{\mu}} \left(x_i - \mu_{x_i} \right)\left(x_j - \mu_{x_j} \right) .$$

.

$$\mu_y =$$

$$\int_{-\infty}^{\infty}\left[g(\underline{\mu})+\sum_i\left(\frac{\partial g}{\partial x_i}\right)_{\underline{x}=\underline{\mu}}\left(x_i-\mu_{x_i}\right)+\frac{1}{2}\sum_i\sum_j\left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right)_{\underline{x}=\underline{\mu}}\left(x_i-\mu_{x_i}\right)\left(x_j-\mu_{x_j}\right)\right]\left(\prod_{i=1}^{n} f_i(x_i)dx_i\right)$$

Since $E(x_i-\mu_{x_i})=0$ and $E((x_i-\mu_{x_i})^2)=\sigma_{x_i}^2$ then the second term in the square brackets is zero and only those terms with $j=i$ contribute in the third term. Thus,

$$\mu_y = g(\underline{\mu})+\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial^2 g}{\partial x_i^2}\right)_{\underline{x}=\underline{\mu}}\sigma_{x_i}^2$$

$$\sigma_y^2 =$$

$$\int_{-\infty}^{\infty}\left[\sum_i\left(\frac{\partial g}{\partial x_i}\right)_{\underline{x}=\underline{\mu}}\left(x_i-\mu_{x_i}\right)+\frac{1}{2}\sum_i\sum_j\left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right)_{\underline{x}=\underline{\mu}}\left(x_i-\mu_{x_i}\right)\left(x_j-\mu_{x_j}\right)-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial^2 g}{\partial x_i^2}\right)_{\underline{x}=\underline{\mu}}\sigma_{x_i}^2\right]^2\left(\prod_{i=1}^{n} f_i(x_i)dx_i\right)$$

$$\sigma_y^2 = \int_{-\infty}^{\infty} \left[\sum_i \sum_j \left(\frac{\partial g}{\partial x_i} \right)_{\underline{x}=\underline{\mu}} \left(\frac{\partial g}{\partial x_j} \right)_{\underline{x}=\underline{\mu}} \left(x_i - \mu_{x_i} \right) \left(x_j - \mu_{x_j} \right) + \ldots \right]$$

$$\left[\ldots \sum_i \sum_j \sum_k \left(\frac{\partial g}{\partial x_k} \right)_{\underline{x}=\underline{\mu}} \left(\frac{\partial^2 g}{\partial x_i \partial x_j} \right)_{\underline{x}=\underline{\mu}} \left(x_i - \mu_{x_i} \right) \left(x_j - \mu_{x_j} \right) \left(x_k - \mu_{x_k} \right) - \sum_i \sum_j \left(\frac{\partial^2 g}{\partial x_i^2} \right) \left(\frac{\partial g}{\partial x_j} \right) \sigma_{x_i}^2 \left(x_j - \mu_{x_j} \right) + \ldots \right]$$

$$\left[\ldots \frac{1}{4} \sum_i \sum_j \sum_k \sum_\ell \left(\frac{\partial^2 g}{\partial x_i \partial x_j} \right)_{\underline{x}=\underline{\mu}} \left(\frac{\partial^2 g}{\partial x_k \partial x_\ell} \right)_{\underline{x}=\underline{\mu}} \left(x_i - \mu_{x_i} \right) \left(x_j - \mu_{x_j} \right) \left(x_k - \mu_{x_k} \right) \left(x_\ell - \mu_{x_\ell} \right) \ldots \right]$$

$$\left[\ldots - \frac{1}{2} \sum_i \sum_j \sum_k \left(\frac{\partial^2 g}{\partial x_i \partial x_j} \right)_{\underline{x}=\underline{\mu}} \left(\frac{\partial^2 g}{\partial x_k^2} \right)_{\underline{x}=\underline{\mu}} \sigma_{x_k}^2 \left(x_i - \mu_{x_i} \right) \left(x_j - \mu_{x_j} \right) + \ldots \right]$$

$$\left[\ldots \frac{1}{4} \sum_i \sum_j \left(\frac{\partial^2 g}{\partial x_i^2} \right)_{\underline{x}=\underline{\mu}} \left(\frac{\partial^2 g}{\partial x_j^2} \right)_{\underline{x}=\underline{\mu}} \sigma_{x_i}^2 \sigma_{x_j}^2 \right] \left(\prod_{i=1}^{n} f_i(x_i) dx_i \right)$$

Since $E(x_i - \mu_{x_i}) = 0$ and $E((x_i - \mu_{x_i})^2) = \sigma_{x_i}^2$ then the first term is zero unless *i=j*, the second term is zero unless *i=j=k*, the third term is zero, the fourth term is zero unless *k=i (j)*, *ℓ=j (i)* and the fifth term is zero unless *i=j*.

Thus

$$\sigma_y^2 = \sum_{i=1}^{n}\left(\frac{\partial g}{\partial x_i}\right)^2_{\underline{x}=\underline{\mu}} \sigma_{x_i}^2 + \sum_i \left(\frac{\partial g}{\partial x_i}\frac{\partial^2 g}{\partial x_i^2}\right)_{\underline{x}=\underline{\mu}} E\left[\left(x_i - \mu_i\right)^3\right] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}\left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right)^2_{\underline{x}=\underline{\mu}} \sigma_{x_i}^2 \sigma_{x_j}^2 +$$

$$\frac{1}{4}\sum_i \left(\frac{\partial^2 g}{\partial x_i^2}\right)^2_{\underline{x}=\underline{\mu}} E\left(\left(x_i - \mu_{x_i}\right)^4 - \sigma_{x_i}^4\right)$$

Note that there are third order Taylor series terms of the form $\left(\frac{\partial g}{\partial x_i}\right)_{\underline{x}=\underline{\mu}}\left(\frac{\partial^3 g}{\partial x_i \partial x_j^2}\right)_{\underline{x}=\underline{\mu}} \sigma_{x_i}^2 \sigma_{x_j}^2$ which are of the same order as terms in the above equation.

APPENDIX D
Mean and Variance for Selected Distributions

Normal distribution

$$f(x) = e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\mu_x = \mu$$

$$\sigma_x^2 = \sigma^2$$

Exponential distribution

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$$

$$\mu_x = \theta$$

$$\sigma_x^2 = \theta^2$$

Binomial distribution

$$f(x;n,\theta) = \frac{n!\theta^{x}(1-\theta)^{n-x}}{x!(n-x)!}$$

$$\mu_x = n\theta$$

$$\sigma_x^2 = n\theta(1-\theta)$$

Chi-squared distribution with ν degrees of freedom

$$f(x) = \frac{x^{\left(\frac{\nu-2}{2}\right)} e^{-\frac{x}{2}}}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)}$$

$$\mu_x = \nu$$

$$\sigma_x^2 = 2\nu$$

Rectangular distribution

$$f(x) = \frac{1}{2a} \qquad -a \le x \le a$$
$$= 0 \qquad otherwise$$

$$\mu_x = 0$$

$$\sigma_x^2 = \frac{a^2}{3}$$

U distribution

$$f(x) = \frac{2n+1}{2a}\left(\frac{x}{a}\right)^{2n} \qquad -a \le x \le a;\ n \text{ an integer greater than } 0$$

$$= 0 \qquad \textit{otherwise}$$

$$\mu_x = 0$$

$$\sigma_x^2 = \left(\frac{2n+1}{2n+3}\right)a^2$$